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The Top of the Lattice of Normal Subgroups of the Grigorchuk Group

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A complete description of the lattice of all normal subgroups not contained in the stabilizer of the fourth level of the tree and, consequently, of index $\leq 2^{12}$ in the Grigorchuk group G is given. This leads to the following sharp version of the *congruence property*: a normal subgroup not contained in the stabilizer at level $n+1$ contains the stabilizer at level $n+3$ (in fact such a normal subgroup contains the subgroup N_{n+1}), but, in general, it does not contain the stabilizer at level $n+2$. The determination of all normal subgroups at each level $n \geq 4$ is then reduced to the analysis of certain G-modules which depend only on n and the previous description, as for the analogous problem for the automorphism group of the regular rooted tree. © 2001 Elsevier Science

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1. INTRODUCTION

The Grigorchuk group G was introduced in [5] and deeply investigated in [6]. We use [7, pp. 164–176; 8, Chap. VIII] as basic references for well known elementary facts and terminology.

Each normal subgroup $N \trianglelefteq G$ has a *level*, namely the largest integer n such that N acts trivially on the first n levels of the rooted binary tree on which G acts naturally. In this paper we determine all normal subgroups at levels $n = 0, 1, 2$, and 3 of G providing a full description of the lattice of all normal subgroups of index $\leq 2^{12}$.

It is well known [7, 8] that G satisfies the *congruence property*: every finite index subgroup of G contains $\mathrm{St}_G(m)$, the stabilizer of level m, for some m; indeed a normal subgroup at level n contains $\mathrm{St}_G(n+6)$. As a consequence of this, G is *just infinite*, namely all its proper quotients are finite, and the set of all its normal subgroups at a given level is finite. As a by-product of our investigations, we show that a normal subgroup at level n contains the subgroup N_{n+1} (Theorem 5.12), where $N_{n+1} = N_1 \times \cdots \times N_1$ (2^n times) and N_1 is the third term of the lower central series of G. This gives a sharp version of the congruence property: a normal subgroup N at level n contains $\mathrm{St}_G(n+3)$ (Corollary 5.13) but, in general, does not contain $\mathrm{St}_G(n+2)$ (Remark 5.14).

In Section 2 we briefly recall some notation and preliminaries on G and on some of its normal subgroups which play a fundamental role in the sequel (H, B, K, K_n, N_m, $n, m \geq 1$). Each of the subsequent sections is devoted to the determination of normal subgroups at levels $n = 0, 1, 2$, and 3.

The strategy consists in finding, for each level $n = 0, 1$, a suitable normal subgroup L_n containing $[\mathrm{St}_G(n), \mathrm{St}_G(n)]$ and contained in all normal subgroups of G at level n that we shall denote by $J_{n,x}$, $x = 1, 2, 3, \ldots$; these latter correspond to suitable G-submodules of $\mathrm{St}_G(n)/L_n$. The situation is more complicated for levels 2 and 3 where we need a finer analysis still consisting in the determination of suitable G-submodules of different G-modules (see Remarks 5.10 and 6.8.1). The lists of the normal subgroups at each level (with their index in G) are then presented in the tables, where, for each subgroup, the corresponding submodule is also included, although this shall be defined only later, during the proofs. We also include a picture with the complete lattice of all normal subgroups of index $\leq 2^8$.

Regarding the analysis of the third level, the proofs of some statements are only sketched, in the sense that elementary tedious computations leading to commutator inclusions, or description of the G-actions on very small submodules, are omitted.

We stop this analysis at the third level because, as shown in the last section, the determination of all normal subgroups at each level $n \geq 4$ is reduced to the analysis of all G-submodules of certain G-modules whose structure only depend on n and on suitable normal subgroups at level 3. This is very close to the analogous problem for the automorphism group of the regular rooted tree discussed in [4]: however, as remarked there, the completion of this analysis requires a more developed theory for the G-actions on abelian groups.

2. NOTATION AND PRELIMINARIES

2.1. We adopt the following notation for conjugacy and commutator for elements x, y in a group Γ:

$$x^y = yxy^{-1}, \qquad [x, y] = xyx^{-1}y^{-1}.$$

Also, for a subset X of Γ, we denote by $\langle X\rangle^\Gamma$ its *normal closure*, while, for a (normal) subgroup $N \leq \Gamma$, $\equiv_N$ denotes *congruence modulo* N. Finally C_n denotes the *cyclic group* of order n.

2.2. Let T be the infinite rooted binary tree and denote by $\mathrm{Aut}(T)$ its automorphism group. We identify the set of vertices of T with the set Σ^* of all (finite) words over the alphabet $\Sigma = \{0, 1\}$. Given $\sigma \in \Sigma^*$, we clearly have $T_\sigma := \sigma\Sigma^* \cong T$ and $\mathrm{Aut}(T_\sigma) \cong \mathrm{Aut}(T)$. If $\{A_\sigma : \sigma \in \Sigma^n\}$ are subgroups of $\mathrm{Aut}(T)$, we can form their *geometric product*

$$\prod_{\sigma\in\Sigma^n} A_\sigma = \{(a_{0^n}, a_{0^{n-1}1}, \ldots, a_{1^n}) : a_\sigma \in A_\sigma\},$$

thinking, for every $\sigma \in \Sigma^n$, A_σ acting on the subtree T_σ.

For instance for an element g in $\mathrm{St}_{\mathrm{Aut}(T)}(1)$, the stabilizer of the first level of the tree, we have $g = (g_0, g_1) \in \mathrm{Aut}(T_0) \times \mathrm{Aut}(T_1)$ for unique $g_i \in \mathrm{Aut}(T)$.

In the present paper we consider several geometric products, but we will not introduce a specific notation for this situation; unless otherwise specified, all the products of subgroups of $\mathrm{Aut}(T)$ are geometric.

2.3. The Grigorchuk group G is a finitely generated subgroup of $\mathrm{Aut}(T)$. The first generator a is the automorphism permuting the top two branches of T, namely T_0 and T_1, while the remaining generators are defined recursively as follows: $b = (a, c)$, $c = (a, d)$, $d = (1, b)$.

2.3.1. For a G-module M, we denote by $M \times M$ (interpreted as a *geometric product*) the G-module with the action $(m_1, m_2)^a = (m_2, m_1)$, $(m_1, m_2)^b = (m_1^a, m_2^c)$, etc., $m_i \in M$. Similarly one defines the G-action on $M \times \cdots \times M(2^n$ factors), $n \geq 0$. In particular, if $M = C_2$, with the trivial G-action, $P(n) := C_2 \times \cdots \times C_2(2^n$ factors) is the C_2–valued nth *permutation G-module* (see [3]).

If M_1 and M_2 are G-modules, we define their *direct sum* $M_1 \oplus M_2$ as the module $M_1 \times M_2$ with the *diagonal action* of G: $(m_1, m_2)^g = (m_1^g, m_2^g)$, $m_i \in M_i$, $g \in G$.

2.4. The subgroup $H = \langle b, c, d\rangle^G$ has index $[G : H] = 2$, it is generated (as a subgroup) by the elements b, c, d, b^a, c^a and d^a, and it is also the G-stabilizer of the first level of the tree: $H = \mathrm{St}_G(1)$. We denote by

$$\phi = \phi_0 \times \phi_1 : H \ni g \mapsto \phi(g) = (g_0, g_1) \in G \times G$$

the monomorphism of H into $G \times G$. Note that ϕ_0 and ϕ_1 are surjective.

2.5. Setting

$$\begin{aligned} &x = [a, b] = abab, \qquad y = [b, ada] = badabada = (x, 1), \\ &z = [aba, d] = abadabad = (1, x), \end{aligned} \tag{1}$$

one has the following relations which will be widely used ([8, p. 230], where different notations are used).

LEMMA.

$$\begin{array}{lllllllll} axa & = & x^{-1} & aya & = & z & aza & = & y \\ bxb & = & x^{-1} & byb & = & y^{-1} & bzb & = & x^{-1}z^{-1}x \\ dxd & = & z^{-1}x & dyd & = & y & dzd & = & z^{-1}. \end{array}$$

2.6. LEMMA. $(ca)^4 = (z^{-1}x, y^{-1}x^{-1})x^2$.

Proof. Straightforward verification using $(ca)^4 = ((ad)^2, (da)^2)$ and $x^2 = (ab)^4 = ((ca)^2, (ac)^2)$. ∎

2.7. We recall that $B := \langle b\rangle^G$ denotes the normal closure of b and $D := \langle c, c^a\rangle$ denotes the subgroup generated by c and c^a, which is isomorphic to the dihedral group of order 8. Also $K := \langle x\rangle^G$ is generated by elements x, y, and z, $K_1 := K \times K$, and recursively $K_n := K_{n-1} \times K_{n-1}$ (see [7, 8] for more details).

2.8. Following [2, 7], we set $N_1 := \langle K_1, x^2\rangle$ and $N_m := N_{m-1} \times N_{m-1}$, $m = 2, 3, \ldots$. Using Lemma 2.5 it is easy to check that N_1 (and thus every N_m) is normal in G (actually they belong to the lower central series of G; see [2, 7, 9]). By Lemma 2.6 we have $x^2 \equiv_{K_1} (ca)^4$ so that $N_1 = K_1 \rtimes \langle (ca)^4\rangle \cong K_1 \rtimes C_2$. Moreover it is clear that $[N_1 : K_1] = [K : N_1] = 2$ and $[G : N_1] = 2^5$.

2.9. LEMMA. *Let N be a normal subgroup of G contained in H and $M = \phi_0(N)$. Then*

$$[M, B] \times [M, B] \le N \le M \times M.$$

Proof. It is obvious that $N \leq M \times M$. For the other inclusion note that $\forall v \in M, \exists w \in M : (v, w) \in N$. Then, since $N \trianglelefteq G$ and $B \times B \leq G$, we have, $\forall g \in B$,

$$N \ni [(v, w), (g, 1)] = ([v, g], 1)$$

so that $[M, B] \times \{1\} \leq N$; since N is a-invariant, $[M, B] \times [M, B] \subseteq N$. ∎

3. THE NORMAL SUBGROUPS OF G NOT CONTAINED IN H

3.1. We have the following structure decomposition: $G = (B \times B) \rtimes (D \rtimes \langle a \rangle)$, where $D = \langle c, c^a \rangle$. Indeed, from $G = H \rtimes \langle a \rangle$ and $H = (B \times B) \rtimes D$ [7, p. 167; 8, p. 229] we have $G = (B \times B)D\langle a \rangle$. Moreover $B \times B$ is normal in G, D and $\langle a \rangle$ form a semidirect product, and $(B \times B) \cap (D \rtimes \langle a \rangle) = \{1\}$.

3.2. The easy proof of the following lemma is omitted.

LEMMA. *The following is the list of all normal subgroups of $Q = D \rtimes \langle a \rangle$ not contained in D:*

$$Q, \ \langle caca \rangle \rtimes \langle a \rangle \cong C_4 \rtimes C_2 \qquad \text{and} \qquad \langle ca \rangle \cong C_8.$$

3.3. LEMMA. *Let N be a normal subgroup of G not in H. Then $[G : N] \leq 4$. In particular $N \geq [G, G]$.*

Proof.

Step 1: $N \geq [N, B \times B] \geq [B \times B, B \times B]$. Let $g = ha$, with $h \in H$, be an element of N not in H. If $h = (g_0, g_1)$, $\xi \in B$, and $f = (\xi^{-1}, 1)$ we have

$$[g, f] = (\xi, \ g_1 \xi^{-1} g_1^{-1}).$$

Now let $\eta \in B$ and set $l = (\eta, 1)$. Then we have

$$[[g, f], l] = ([\xi, \eta], 1).$$

This shows that $[B, B] \times \{1\} \leq [N, B \times B]$. Since N is a-invariant we obtain the proof of Step 1.

Step 2: $N \geq \langle K_1, (b, b) \rangle$. Denote by P the projection of N into $Q = D \rtimes \langle a \rangle$, i.e., $P = \{q \in Q : \exists \gamma \in B \times B \text{ s.t. } \gamma q \in N\}$. P is normal in Q and is not contained in D. By Lemma 3.2 there exists $h \in \{1, c\}$ s.t. $ah \in P$ and thus $\exists \gamma \in B \times B$ s.t. $\beta := \gamma a h \in N$. Then, by Step 1,

$$\begin{aligned}[ah, (b, 1)] = [\gamma^{-1}\beta, (b, 1)] &= [\beta, (b, 1)]^{\gamma^{-1}} [\gamma^{-1}, (b, 1)] \\ &\in N[B \times B, B \times B] = N.\end{aligned}$$

Now $[ah, (b, 1)] = (b, t_h)$, where $t_h = b$ if $h = 1$ and $t_h = aba$ if $h = c$. Then

$$N \ni (c, a)(b, t_h)(c, a)(b, t_h) = (1, x^{\pm 1}),$$

where we have $+1$ if $h = 1$ and -1 if $h = c$. Finally

$$\langle (1, x^{-1}), (b, aba)\rangle^G = \langle (1, x), (b, b)\rangle^G = \langle K_1, (b, b)\rangle$$

and this proves Step 2.

Step 3: Conclusion. From the previous step it follows that N corresponds to a submodule $\overline{N} := N/\langle K_1, (b, b)\rangle$ of $(B \times B) \rtimes P/\langle K_1, (b, b)\rangle = C_2 \times P$. But $\overline{N}$ projects onto P, so that $[(B \times B) \rtimes P : N] \leq 2$. Moreover from 3.1 and Lemma 3.2 it follows that $[G : (B \times B) \rtimes P] \leq 2$. The proof of the lemma is now complete. ∎

3.4. *Remark.* Step 1 is well known [7, p. 149; 8, pp. 239–240]; more generally, one can show that if $N \leq \mathrm{St}_G(n)$ but $N \not\leq \mathrm{St}_G(n+1)$, $n \geq 1$, then $N \geq [N, K_{n+1}] \geq [K_{n+1}, K_{n+1}] \geq K_{n+3} \geq \mathrm{St}_G(n+6)$. This is an important estimate which will be improved in 5.12 and 5.13.

3.5. THEOREM. *The following are all the normal subgroups of G not contained in H:*

Subgroup	Description	Generators	Index	Submodule of G^{ab}
$J_{0,1}$	G	b, c, a	1	G^{ab}
$J_{0,2}$	$\langle b, a, [G, G]\rangle \equiv \langle a, b\rangle^G$	b, a, a^c	2	$\langle\bar{b}\rangle \times \langle\bar{a}\rangle$
$J_{0,3}$	$\langle c, a, [G, G]\rangle \equiv \langle a, c\rangle^G$	c, a, a^d	2	$\langle\bar{c}\rangle \times \langle\bar{a}\rangle$
$J_{0,4}$	$\langle d, a, [G, G]\rangle \equiv \langle a, d\rangle^G$	d, a, a^b	2	$\langle\bar{b}\bar{c}\rangle \times \langle\bar{a}\rangle$
$J_{0,5}$	$\langle b, ca, [G, G]\rangle \equiv \langle b, ac\rangle^G$	b, ac	2	$\langle\bar{b}\rangle \times \langle\bar{c}\bar{a}\rangle$
$J_{0,6}$	$\langle c, ba, [G, G]\rangle \equiv \langle c, ab\rangle^G$	c, ab	2	$\langle\bar{c}\rangle \times \langle\bar{b}\bar{a}\rangle$
$J_{0,7}$	$\langle ba, ca, [G, G]\rangle \equiv \langle d, ab\rangle^G$	d, ab	2	$\langle\bar{b}\bar{a}\rangle \times \langle\bar{c}\bar{a}\rangle$
$J_{0,8}$	$\langle a, [G, G]\rangle \equiv \langle a\rangle^G$	a, a^b, a^c, a^d	4	$\langle\bar{a}\rangle$
$J_{0,9}$	$\langle ba, [G, G]\rangle \equiv \langle ba\rangle^G$	ba, dac	4	$\langle\bar{b}\bar{a}\rangle$
$J_{0,10}$	$\langle ca, [G, G]\rangle \equiv \langle ca\rangle^G$	ca, dab	4	$\langle\bar{c}\bar{a}\rangle$
$J_{0,11}$	$\langle da, [G, G]\rangle \equiv \langle da\rangle^G$	da, bac	4	$\langle\bar{b}\bar{c}\bar{a}\rangle$

Proof. Set $\bar{g} = g \cdot [G, G]$ for $g \in G$; the subgroups are in one to one correspondence with the submodules of $G^{ab} = \langle\bar{b}\rangle \times \langle\bar{c}\rangle \times \langle\bar{a}\rangle \cong C_2 \times C_2 \times C_2$ not contained in $\langle\bar{b}\rangle \times \langle\bar{c}\rangle$, as listed above. ∎

4. THE NORMAL SUBGROUPS OF G IN $H = \mathrm{St}_G(1)$ BUT NOT IN $\mathrm{St}_G(2)$

4.1. LEMMA. *Let N be a normal subgroup of G contained in H but not in $\mathrm{St}_G(2)$. Then $N \geq N_1$.*

Proof.
Step 1: $N \geq [B \times B, B \times B]$. Indeed $M := \phi_0(N)$ is normal in G and not contained in H. From Lemma 3.3. it follows that $K \leq [G, G] \leq M$, so that, by Lemma 2.9,

$$N \geq [M, B] \times [M, B] \geq [K, B] \times [K, B] = [B, B] \times [B, B].$$

Step 2: $N \geq K_1$. Recalling from 3.1 that $H = (B \times B) \rtimes D$, let $P = \{q \in D : \exists\, \gamma \in B \times B \text{ s.t. } \gamma q \in N\}$ be the projection of N into D. Then $P = D$ or $P = \langle caca \rangle$ so that there exists $\gamma \in B \times B$ s.t. $\beta := \gamma caca \in N$. Therefore, by the previous step

$$\begin{aligned}(x, 1) = [caca, (b, 1)] = [\gamma^{-1}\beta, (b, 1)] = [\beta, (b, 1)]^{\gamma^{-1}} \cdot [\gamma^{-1}, (b, 1)] \\ \in N \cdot [B \times B, B \times B] = N\end{aligned}$$

and this proves the step.
Step 3: Conclusion. Let γ be as in the previous step. Then

$$\gamma caca \gamma caca = \gamma \cdot \gamma^{caca}(ca)^4 \in N.$$

But $((B \times B) \rtimes D)/K_1 = (C_2 \times C_2) \times D$, so that $\gamma \cdot \gamma^{caca} \in K_1 \subseteq N$ and this ends the proof of the lemma. ∎

4.2. THEOREM. *The following are all the normal subgroups of G contained in H but not in* $\mathrm{St}_G(2)$:

Subgroup	Description	Generators	Index	Submodule of H/N_1
$J_{1,1}$	$\langle N_1, c, aca\rangle \equiv \langle c\rangle^G$	c, c^a, c^{ad}, c^{ada}	8	$\langle \bar{c}, \overline{aca}\rangle$
$J_{1,2}$	$\langle N_1, (b,1), (1,b), c, aca\rangle \equiv H$	c, c^a, d, d^a	2	$\langle \bar{c}, \overline{aca}, \overline{(b,1)}, \overline{(1,b)}\rangle$
$J_{1,3}$	$\langle N_1, (b,b), c, aca\rangle$	c, c^a, dd^a	4	$\langle \bar{c}, \overline{aca}, \overline{(b,b)}\rangle$
$J_{1,4}$	$\langle N_1, (b,1)c, (1,b)aca\rangle \equiv \langle d^a c\rangle^G$	$d^a c, dc^a$	8	$\langle \overline{(b,1)}\bar{c}, \overline{(1,b)}\overline{aca}\rangle$
$J_{1,5}$	$\langle B, (b,b)\rangle$	$d^a c, dc^a, dd^a$	4	$\langle \overline{(b,1)}\bar{c}, \overline{(1,b)}\overline{aca}, \overline{(b,b)}\rangle$
$J_{1,6}$	$\langle N_1, (1,b)c, (b,1)aca\rangle \equiv B$	b, b^a, b^{ad}, b^{ada}	8	$\langle \overline{(1,b)}\bar{c}, \overline{(b,1)}\overline{aca}\rangle$
$J_{1,7}$	$\langle N_1, (b,b)c, (b,b)aca\rangle \equiv \langle dd^a c\rangle^G$	$dd^a c, dd^a c^a$	8	$\langle \overline{(b,b)}\bar{c}, \overline{(b,b)}\overline{aca}\rangle$
$J_{1,8}$	$\langle N_1, caca\rangle \equiv \langle cc^a\rangle^G$	cc^a, y, z	16	$\langle \bar{c} \cdot \overline{aca}\rangle$
$J_{1,9}$	$\langle N_1, caca, (b,1), (1,b)\rangle$	d, d^a, cc^a	4	$\langle \bar{c} \cdot \overline{aca}, \overline{(1,b)}, \overline{(b,1)}\rangle$
$J_{1,10}$	$\langle N_1, caca, (b,b)\rangle \equiv [G,G]$	$y, cc^a, d^a d$	8	$\langle \bar{c} \cdot \overline{aca}, \overline{(b,b)}\rangle$
$J_{1,11}$	$\langle N_1, (b,1)caca, (b,b)\rangle \equiv \langle d^a cc^a\rangle^G$	$y, z, d^a cc^a, dd^a$	8	$\langle \overline{(b,1)}\bar{c} \cdot \overline{aca}, \overline{(b,b)}\rangle$
$J_{1,12}$	$\langle N_1, (b,b)caca\rangle \equiv K$	x, y, z	16	$\langle \overline{(b,b)}\bar{c} \cdot \overline{aca}\rangle$

Proof. From the previous lemma, N corresponds to an a-invariant submodule of

$$\begin{aligned}\frac{H}{N_1} = \frac{(B \times B) \rtimes D}{N_1} = \frac{((B \times B) \rtimes D)/K_1}{N_1/K_1} &= \langle \overline{(1,b)}, \overline{(b,1)}, \bar{c}, \overline{aca}\rangle \\ &= C_2 \times C_2 \times C_2 \times C_2\end{aligned}$$

(where $\bar{g} := gN_1$ for any $g \in G$) not contained in $\mathrm{St}_G(2)/N_1 = C_2 \times C_2 \times \{1\} \times \{1\}$ (this equality is obvious, if one notes that $N_1 \leq \mathrm{St}_G(2) \leq H$). These submodules are listed in the statement. ∎

4.2.1. *Remark.* From the analysis in the last proof (see also [3]), it follows that $\mathrm{St}_G(2) = (B \times B) \rtimes \langle (ca)^4 \rangle$.

5. THE NORMAL SUBGROUPS OF G IN $\mathrm{St}_G(2)$ BUT NOT IN $\mathrm{St}_G(3)$

5.1. In this section N will denote a normal subgroup of G contained in $\mathrm{St}_G(2)$ but not in $\mathrm{St}_G(3)$ and M will denote the subgroup $\phi_0(N)$. Clearly M is a normal subgroup of G contained in $\mathrm{St}_G(1) = H$ but not in $\mathrm{St}_G(2)$. Moreover $M = \phi_1(N)$ and $N \leq M \times M$, but $M \times M$ may not be contained in G.

5.2. PROPOSITION. *M is necessarily one of the subgroups $J_{1,4}$, $J_{1,5}$, $J_{1,6}$, $J_{1,8}$, $J_{1,10}$, $J_{1,12}$.*

Proof. Clearly $M \leq \phi_0(\mathrm{St}_G(2)) = \langle B, adad \rangle = J_{1,5}$. Then, using the list given in Theorem 4.2 (see also Figure 1), the proof can be easily completed. ∎

5.3. LEMMA. *We have $[B, B] = \langle N_2, x^2 \rangle$ and $B^{ab} = C_2 \times C_2 \times C_2 \times C_2$.*

Proof. It is easy to check that $\langle N_2, x^2 \rangle$ is normal in G and since $x^4 = ((ca)^4, (ca)^4) \equiv_{K_2} (x^2, x^2) \in N_2$ (by Lemma 2.6.) we have

$$[B : \langle N_2, x^2 \rangle] = \frac{[B : N_2]}{[\langle N_2, x^2 \rangle : N_2]} = \frac{2^5}{2} = 2^4.$$

But B is generated by the four involutions [7, p. 166; 8, p. 227] b, $aba = xb$, $dabad = abaz$, $adabada = by$, and their commutators belong to $\langle N_2, x^2 \rangle$, as one checks immediately; this concludes the proof. ∎

5.4. LEMMA. *If $M \neq J_{1,8}$ then $N \geq N_2$.*

Proof. If $M = J_{1,4}, J_{1,5}$ or $J_{1,10}$, it is easy to check that $[M, B] \geq N_1$ so that, by Lemma 2.9, $N \geq N_2$. Now suppose that $M = J_{1,6} \equiv B$ or $M = J_{1,12} \equiv K$. In both cases $[B, M] = [B, B]$ and, by Lemma 2.9,

$$[B, B] \times [B, B] \leq N \leq B \times B.$$

Since $x^2 \in [B, B]$, it remains to prove that $(y, 1) \in N$. But $x \in M$, so that there exists $s \in B$ s.t. $(x, s) \in N$. Denoting $\bar{g} = g[B, B]$, it follows that $\overline{N} := N/([B, B] \times [B, B])$ contains $w := (\bar{x}, \bar{s})^b (\bar{x}, \bar{s})^{baba} = (\bar{y}, \bar{t})$, where

$\bar{t} \in \langle \bar{x}\bar{y}, \bar{z}\bar{y}\rangle$ as easily follows from Lemmas 2.5 and 5.3. If $\bar{t} = \bar{y}\bar{z}$, then $\overline{N} \ni w(w^b w)^a = (\bar{y}, 1)$. If $\bar{t} = \bar{x}\bar{y}$ or $\bar{t} = \bar{x}\bar{z}$ then $\overline{N} \ni ww^{ab}w^{aba}w^{ba} = (\bar{y}, 1)$ so that, in all cases, $(y, 1) \in N$. ∎

5.5. THEOREM. *If N contains N_2 then it is equal to one of the following subgroups*:

Subgroup	Description	Index	Submodule of $\mathrm{St}_G(2)/N_2$
$J_{2,1}$	$(B \times B)\rtimes C_2 \equiv \mathrm{St}_G(2)$	2^3	$(C_2 \times C_2 \times C_2 \times C_2) \times C_2$
$J_{2,2}$	$\langle K_1, (b, 1)(ca)^4, (1, b)(ca)^4\rangle$	2^4	$\langle(\alpha, 1)\epsilon, (\beta, 1)\epsilon, (1, \alpha)\epsilon, (1, \beta)\epsilon\rangle$
$J_{2,3}$	$B \times B$	2^4	$C_2^2 \times C_2^2$
$J_{2,4}$	$\langle K_1, (b, b), (ca)^4\rangle$	2^4	$\langle(\alpha, \beta), (\beta, \alpha), (\beta, \beta), \epsilon\rangle$
$J_{2,5}$	$\langle K_1, (b, b)(ca)^4\rangle$	2^5	$\langle(\alpha, \beta)\epsilon, (\beta, \alpha)\epsilon, (\beta, \beta)\epsilon\rangle$
$J_{2,6}$	$\langle K_1, (b, b)\rangle$	2^5	$\langle(\alpha, \beta), (\beta, \alpha), (\beta, \beta)\rangle$
$J_{2,7}$	$\langle K_1, (ca)^4\rangle \equiv N_1$	2^5	$\langle(\alpha\beta, 1), (1, \alpha\beta), \epsilon\rangle$
$J_{2,8}$	$\langle N_2, (x, 1)(ca)^4, (1, x)(ca)^4\rangle$	2^6	$\langle(\alpha\beta, 1)\epsilon, (1, \alpha\beta)\epsilon\rangle$
$J_{2,9}$	$\langle N_2, (x, 1), (1, x)\rangle \equiv K_1$	2^6	$\langle(\alpha\beta, 1), (1, \alpha\beta)\rangle$
$J_{2,10}$	$\langle N_2, (x, x), (ca)^4\rangle$	2^6	$\langle(\alpha\beta, \alpha\beta), \epsilon\rangle$
$J_{2,11}$	$\langle N_2, (x, x)(ca)^4\rangle$	2^7	$\langle(\alpha\beta, \alpha\beta)\epsilon\rangle$
$J_{2,12}$	$\langle N_2, (x, x)\rangle$	2^7	$\langle(\alpha\beta, \alpha\beta)\rangle$

Proof. N corresponds to a G-invariant submodule of (see 2.3.1 and Remark 4.2.1)

$$\begin{aligned}
\mathrm{St}_G(2)/N_2 &= \frac{(B \times B)\rtimes C_2}{N_2} \\
&= \langle(\alpha, 1), (\beta, 1)\rangle\langle(1, \alpha), (1, \beta)\rangle \times \langle\epsilon\rangle \\
&= P(2) \oplus P(0),
\end{aligned} \tag{2}$$

where α, β, and ϵ denote the images in (2) of aba, b, and $(adad, dada) \equiv (ca)^4$, respectively.

As for the G-module structure of (2) this is a simple verification: the only nontrivial used facts are:

- $b\epsilon b = \epsilon$: it follows from the identity $b(ca)^4 b = (1, y)(ca)^4 \equiv (ca)^4 \mod K_2$;
- $d(1, \alpha)d = (1, \alpha)$: it follows from the identity $d(1, aba)d = (1, x^{-2})(1, aba) \equiv (1, aba) \mod N_2$;
- $b(1, \alpha)b = (1, \alpha)$: it follows from the identity $b(1, aba)b = (1, bz^{-1}bx^{-2})(1, aba) \equiv (1, aba) \mod N_2$.

The G-submodules of $\mathrm{St}_G(2)/N_2$ not in $\mathrm{St}_G(3)/N_2 (= \langle\epsilon\rangle$, as one can check) are those in the list. ∎

5.5.1. *Remark.* From the analysis in the last proof (see also [3]) it follows that $\mathrm{St}_G(3) = N_2 \rtimes \langle (ca)^4 \rangle$.

5.6. We are now left to the determination of the N's not containing N_2. In this case, by Proposition 5.2. and Lemma 5.4, M is necessarily equal to $J_{1,8}$. We start our analysis with a lemma on $J_{1,8} \times J_{1,8}$.

LEMMA. *With the notation of Theorems* 4.2 *and* 5.5 *we have*

$$(J_{1,8} \times J_{1,8}) \cap G = J_{2,11} = K_2 \rtimes \langle (caca, caca), (1, (ca)^4) \rangle \cong K_2 \rtimes (C_4 \times C_2).$$

Proof. Since $(1, caca) \notin G$, the intersection in the statement is a proper subgroup of $J_{1,8} \times J_{1,8} = K_2 \rtimes \langle (caca, 1), (1, caca) \rangle$. To end the proof it then suffices to show that the second equality holds. Since $J_{2,11} = \langle K_2, ((ca)^4, 1), (1, (ca)^4), (ca)^4(x, x) \rangle$ this latter is a consequence of the identity $adadx = z^{adad} acac \equiv acac \bmod K_1$. ∎

5.7. LEMMA. *With the notation of Theorem* 5.5, *if* $M = J_{1,8}$, *then* $N \geq J_{2,8} \times J_{2,8}$.

Proof. It is easy to verify that $[B, J_{1,8}] \geq J_{2,8}$; then apply Lemma 2.9. ∎

5.8. Setting $\gamma := (caca, acac)(J_{2,8} \times J_{2,8})$ and $\delta := (1, (ca)^4)(J_{2,8} \times J_{2,8})$ we have

LEMMA.

$$\frac{J_{2,11}}{J_{2,8} \times J_{2,8}} = \langle \gamma, \delta \rangle = C_4 \times C_2.$$

Proof. First of all, from Theorems 4.2 and 5.5 it follows that $[J_{1,8} : J_{2,8}] = 4$. Moreover, $(acac)^2 \notin J_{2,8}$. Therefore $J_{1,8}/J_{2,8} = C_4 = \langle cacaJ_{2,8} \rangle$. Then the lemma follows from Lemma 5.6. ∎

5.9. Now we complete the list of all normal subgroups in G contained in $\mathrm{St}_G(2)$ but not in $\mathrm{St}_G(3)$.

THEOREM. *If N does not contain N_2, then N is one of the following subgroups* (*with γ and δ as in* 5.8):

Subgroup	Description	Index	Submodule of $J_{2,11}/(J_{2,8} \times J_{2,8})$
$J_{2,13}$	$\langle J_{2,8} \times J_{2,8}, (caca, acac) \rangle$	2^8	$\langle \gamma \rangle$
$J_{2,14}$	$\langle J_{2,8} \times J_{2,8}, (caca, caca) \rangle$	2^8	$\langle \gamma\delta \rangle$

Proof. By Lemma 5.4, $M = J_{1,8}$. Then, by virtue of Lemmas 5.6, 5.7, and 5.8, N corresponds to a G-invariant submodule $\overline{N}$ of $\langle \gamma, \delta \rangle = C_4 \times C_2$. But $caca \in M$, so that N contains an element of the form $(caca, t)$ and $\overline{N}$ contains γ or $\gamma\delta$. But $\langle \gamma \rangle$ and $\langle \gamma\delta \rangle$ are of index 2 in $C_4 \times C_2$ and are G-invariant, as one easily checks. ∎

5.10. *Remark.* Note that there is no G-module from which one can extract all the subgroups of this level. Indeed, if $L \trianglelefteq G$ and $\mathrm{St}_G(2)/L$ is commutative, then $[(ca)^4, (b, 1)] = (y^{-1}, 1) \in L$ and $L \geq K_2$, but $J_{2,13}$ and $J_{2,14}$ do not contain K_2. (Actually one can show that $[\mathrm{St}_G(2), \mathrm{St}_G(2)] = N_2$.)

5.11. LEMMA. $[K, J_{2,h}] \geq N_3$, $h = 1, 2, \ldots, 14$.

Proof. Simple calculations (note that $(y, y), (yz, 1) \in J_{2,h}$ for all h's, $[y, (y, y)] = (z^{-1}, 1, 1, 1)$ and $[x, (yz, 1)] = (x^{-2}, z^{-1}, 1, 1)$). ∎

5.12. THEOREM. *Let N be a normal subgroup of G in $\mathrm{St}_G(m)$ but not in $\mathrm{St}_G(m+1)$. Then $N \geq N_{m+1}$.*

Proof. For $m = 0, 1, 2$, even stronger results were obtained previously. Now suppose $m \geq 3$. Clearly $M := \phi_0^{m-2}(N)$ is a normal subgroup of G contained in $\mathrm{St}_G(2)$ but not in $\mathrm{St}_G(3)$. Therefore $M = J_{2,h}$ for some h. But if $u \in M$, there exist $u_1, \ldots, u_{2^{m-2}-1} \in G$ such that $(u, u_1, \ldots, u_{2^{m-2}-1}) \in N$. Therefore, for $k \in K$,

$$N \ni [(u, u_1, \ldots, u_{2^{m-2}-1}), (k, 1, \ldots, 1)] = ([u, k], 1, \ldots, 1)$$

and applying the previous lemma we obtain

$$N \geq \underbrace{[K, M] \times \cdots \times [K, M]}_{2^{m-2}} \geq \underbrace{N_3 \times \cdots \times N_3}_{2^{m-2}} = N_{m+1}.$$ ∎

5.13. COROLLARY. *If N is as in the previous theorem, then $N \geq \mathrm{St}_G(m+3)$.*

Proof. Since $\mathrm{St}_G(3) \leq N_1$ (see Remark 5.5.1) we have

$$\mathrm{St}_G(m+3) \leq \underbrace{\mathrm{St}_G(3) \times \cdots \times \mathrm{St}_G(3)}_{2^m} \leq \underbrace{N_1 \times \cdots \times N_1}_{2^m} = N_{m+1} \leq N.$$

The first inclusion is in fact an equality [3]: indeed, by the above, the second term is contained in G. ∎

5.14 *Remark.* It is not possible to replace, in the statements of Theorem 5.12 and Corollary 5.13, K_{m+1}, N_{m+1}, and $\mathrm{St}_G(m+3)$ by K_m, N_m, and $\mathrm{St}_G(m+2)$, respectively. To see this consider, for $m = 2$, $N = J_{2,13}$ (or $J_{2,14}$), while, for $m > 2$ one can use

$$N = \underbrace{J_{2,13} \times \cdots \times J_{2,13}}_{2^{m-2}}$$

(which is a subgroup of G since $J_{2,13} \leq K$): the details can be easily checked.

6. THE NORMAL SUBGROUPS OF G IN $\mathrm{St}_G(3)$ BUT NOT IN $\mathrm{St}_G(4)$

6.1. In this section N will denote a normal subgroup of G contained in $\mathrm{St}_G(3)$ but not in $\mathrm{St}_G(4)$ and M will denote the subgroup $\phi_0(N)$.

6.2. PROPOSITION. *M is necessarily one of the groups $J_{2,i}$, with $i = 4, 5, \ldots, 14$.*

Proof. It is an immediate consequence of the fact that (see Remark 5.5.1)

$$\phi_0(\mathrm{St}_G(3)) = N_1 \rtimes \langle adad \rangle = \langle K_1, (ca)^4, (b, b) \rangle = J_{2,4}. \quad \blacksquare$$

6.3. PROPOSITION. *For $i = 1, 2, \ldots, 14$ we have $[B, J_{2,i}] \geq J_{2,12} \times J_{2,12}$.*

Proof. Easy computations: compare with 5.11. $\blacksquare$

6.4. The determination of the set of these N's will be achieved by partitioning it into three families.

The first family consists of those subgroups whose M is equal to either $J_{2,4}$, $J_{2,5}$, or $J_{2,6}$; equivalently, it consists of all N's that have a nontrivial projection onto the C_2 in $\mathrm{St}_G(3) = N_2 \rtimes C_2$.

LEMMA. *If $M = J_{2,i}$, $i = 4, 5, 6$, then $N \geq K_2$.*

Proof. One can check directly that $[M, B] \geq K_1$, so that, by Lemma 2.9, $N \geq K_2$. $\blacksquare$

6.5. PROPOSITION. $\mathrm{St}_G(3)/K_2 = \langle (\bar{x}^2, 1), (1, \bar{x}^2), \epsilon \rangle = (C_2 \times C_2) \times C_2 = P(1) \oplus P(0)$, *where* $(\bar{x}^2, 1)$, $(1, \bar{x}^2)$, *and* ϵ *are the images of* $(x^2, 1)$, $(1, x^2)$, *and* $(ca)^4$, *respectively.*

Proof. Having in mind Remark 5.5.1, this statement follows from $N_1/K_1 = \langle x^2 K_1 \rangle = C_2$ and the fact that ϵ acts trivially on $(C_2 \times C_2)$: indeed, by Lemma 2.5., $adadx^2dada \equiv x^2 \bmod N_2$. As for the G-module structure, these are simple verifications: the only nontrivial relation used is $b(ac)^4 b = (1, y)(ac)^4 \equiv (ac)^4 \bmod K_2$. $\blacksquare$

6.6. THEOREM. *If $M = J_{2,i}$, $i = 4, 5$ or 6 (i.e., if $N \not\leq N_2$), then N is one of the following subgroups*:

Subgroup	Description	Index	Submodule of $\mathrm{St}_G(3)/K_2$
$J_{3,1}$	$\mathrm{St}_G(3) = N_2 \rtimes \langle (ca)^4 \rangle$	2^7	$(C_2 \times C_2) \times C_2$
$J_{3,2}$	$\langle K_2, (x^2, 1)(ca)^4, (1, x^2)(ca)^4 \rangle$	2^8	$\langle (\bar{x}^2, 1)\epsilon, (1, \bar{x}^2)\epsilon \rangle$
$J_{3,3}$	$\langle K_2, (ca)^4, (x^2, x^2) \rangle$	2^8	$\langle \epsilon, (\bar{x}^2, \bar{x}^2) \rangle$
$J_{3,4}$	$\langle K_2, (x^2, x^2)(ca)^4 \rangle$	2^9	$\langle (\bar{x}^2, \bar{x}^2)\epsilon \rangle$
$J_{3,5}$	$\langle K_2, (ca)^4 \rangle$	2^9	$\langle \epsilon \rangle$

Proof. The subgroups N as in the statement correspond bijectively to the G-invariant submodules of $\mathrm{St}_G(3)/K_2 = (C_2 \times C_2) \times C_2 = P(1) \oplus P(0)$ not contained in $(C_2 \times C_2) = P(1)$, as listed in the above table. ∎

6.7. PROPOSITION [7, p. 172; 9].

$$N_1/N_2 = \langle (x,1)N_2, (1,x)N_2 \rangle \times \langle (ca)^4 N_2 \rangle = (C_2 \times C_2) \times C_2 = P(1) \oplus P(0).$$

6.8. The normal subgroups of the second family consist of all N's such that $N_3 \leq N \leq N_2$.

THEOREM. *If $N_3 \leq N \leq N_2$, then N is one of the subgroups in the following list*:

Subgroup	Description	Index
$J_{3,6}$	$\langle N_3, (x^2,x^2) \rangle$	2^{13}
$J_{3,7}$	$\langle N_3, (x,x,x,x) \rangle$	2^{13}
$J_{3,8}$	$\langle N_3, (x,x,x,x), (x^2,x^2) \rangle$	2^{12}
$J_{3,9}$	$\langle N_3, (x,x,1,1), (1,1,x,x) \rangle$	2^{12}
$J_{3,10}$	$\langle N_3, (x,x,1,1)(x^2,x^2), (1,1,x,x)(x^2,x^2) \rangle$	2^{12}
$J_{3,11}$	$\langle N_3, (x,x,1,1), (1,1,x,x), (x^2,x^2) \rangle$	2^{11}
$J_{3,12}$	$\langle N_3, (x,1,x,1), (x,1,1,x), (1,x,x,1) \rangle$	2^{11}
$J_{3,13}$	$\langle N_3, (x,1,x,1)(x^2,x^2), (x,1,1,x)(x^2,x^2), (1,x,x,1)(x^2,x^2) \rangle$	2^{11}
$J_{3,14}$	$\langle N_3, (x,1,x,1), (x,1,1,x), (1,x,x,1), (x^2,x^2) \rangle$	2^{10}
$J_{3,15}$	K_2	2^{10}
$J_{3,16}$	$\langle N_3, (x,1,1,1)(x^2,x^2), (1,x,1,1)(x^2,x^2), (1,1,x,1)(x^2,x^2), (1,1,1,x)(x^2,x^2) \rangle$	2^{10}
$J_{3,17}$	$\langle K_2, (x^2,x^2) \rangle$	2^{9}
$J_{3,18}$	$\langle N_3, (x^2,1), (1,x^2) \rangle$	2^{12}
$J_{3,19}$	$\langle N_3, (x,x,x,x)(x^2,1), (x,x,x,x)(1,x^2) \rangle$	2^{12}
$J_{3,20}$	$\langle N_3, (x,x,x,x), (x^2,1), (1,x^2) \rangle$	2^{11}
$J_{3,21}$	$\langle N_3, (x,x,1,1)(1,x^2), (1,1,x,x)(x^2,1) \rangle$	2^{12}
$J_{3,22}$	$\langle N_3, (x,x,x,x), (x^2,x^2), (1,1,x,x)(1,x^2) \rangle$	2^{11}
$J_{3,23}$	$\langle N_3, (x,x,1,1), (1,x^2), (1,1,x,x), (x^2,1) \rangle$	2^{10}
$J_{3,24}$	$\langle N_3, (x,1,x,1), (x,1,1,x), (1,x,x,1), (x^2,1), (1,x^2) \rangle$	2^{9}
$J_{3,25}$	$\langle N_3, (x,x,1,1), (1,1,x,x), (x^2,x^2), (x,1,x,1)(x^2,1) \rangle$	2^{10}
$J_{3,26}$	N_2	2^{8}
$J_{3,27}$	$\langle N_3, (x,x,1,1), (1,1,x,x), (x,1,x,1), (x^2,x^2), (1,1,1,x)(1,x^2) \rangle$	2^{9}
$J_{3,28}$	$\langle N_3, (x,x,1,1), (1,1,x,x), (x,1,1,1)(1,x^2), (1,1,1,x)(x^2,1) \rangle$	2^{10}
$J_{3,29}$	$\langle N_3, (x,x,1,1), (1,1,x,x), (x,1,1,1)(x^2,1), (1,1,1,x)(1,x^2) \rangle$	2^{10}

Sketch of proof. From Proposition 6.7 it follows that

$$N_2/N_3 = [(C_2 \times C_2) \times (C_2 \times C_2)] \times (C_2 \times C_2) = P(2) \oplus P(1).$$

We recall that (see the proof of 5.13) $\mathrm{St}_G(4) = \mathrm{St}_G(3) \times \mathrm{St}_G(3) = N_3 \rtimes (C_2 \times C_2)$ where $C_2 \times C_2 = \langle ((ca)^4, 1), (1, (ca)^4) \rangle$. But from Lemma 2.6,

$(ca)^4 \equiv_{N_2} (x, x)x^2$ so that

$$\mathrm{St}_G(4)/N_3 = \langle (x, x, 1, 1)(x^2, 1)N_3, (1, 1, x, x)(1, x^2)N_3 \rangle. \tag{3}$$

Thus one is left to determine all the G-invariant submodules of N_2/N_3 not contained in (3): these correspond to the subgroups in the list. The details are left to the reader. ∎

6.8.1. *Remark.* $\mathrm{St}_G(3)/N_3$ is commutative, but it is not the direct sum of permutation G-modules, since $C_2 = \langle (ca)^4 N_2 \rangle$ is not G-invariant. For the sake of clarity we have preferred to split the analysis in two theorems, whose proof depends on the analysis of permutation modules as in the previous levels.

6.9. Now we consider the last family consisting of all N's which do not contain N_3. The following lemma, together with Lemma 6.4, restricts the possibilities for the groups $M = \phi_0(N)$ with N belonging to this family.

LEMMA. *If $M = J_{2,i}$, for $i = 7, 8, 9, 10, 12$, then $N \geq N_3$.*

Sketch of proof. We have a case-by-case analysis (keeping in mind Proposition 6.3).

- $M = J_{2,7},\ J_{2,8}$, or $J_{2,9}$. In these cases direct computations show that $[M, B] \geq N_2$ and we can conclude by Lemma 2.9
- $M = J_{2,10}$. First show that $[M, B] \geq J_{3,17}$, so that $N \geq J_{3,17} \times J_{3,17}$. Conclude by analyzing $\overline{M} \times \overline{M}$, where

$$\overline{M} := M/J_{3,17} = \frac{M/K_2}{J_{3,17}/K_2} = C_2 \times C_2 \times C_2 = \langle x^2 J_{3,17}, y^2 J_{3,17}, x^2 yz J_{3,17} \rangle.$$

- $M = J_{2,12}$. First show that $[M, B] \geq J_{3,16}$ and that $\overline{M} := M/J_{3,16} = C_4 \times C_2 = \langle (x, x)J_{3,16} \rangle \times \langle (x^2, 1)J_{3,16} \rangle$. Conclude by analyzing $\overline{M} \times \overline{M}$. ∎

6.10.1. In what follows we denote by $\bar{g}$ the coset $g \cdot J_{2,12},\ \forall g \in G$. In order to prove the next theorem, which concludes the classification of all normal subgroups at level 3, we need some further results. The next lemma is an immediate consequence of Lemma 2.6.

LEMMA. $(ca)^4 \equiv x^2 \bmod J_{2,12}$, *i.e.,* $(\bar{c}\bar{a})^4 = \bar{x}^2$.

6.10.2. Set $A_1 = J_{2,11}/J_{3,27}$ and $A_2 = J_{2,11}/(J_{2,12} \times J_{2,12})$; one easily shows that

$$A_2 = \langle (\bar{y}, 1), (1, \bar{y}), (1, (\bar{c}\bar{a})^4), (\bar{a}\bar{c}\bar{a}\bar{c}, \bar{a}\bar{c}\bar{a}\bar{c}) \rangle = (C_2 \times C_2) \times (C_2 \times C_4). \tag{4}$$

LEMMA. *If $M = J_{2,11}, J_{2,13}$, or $J_{2,14}$, then N contains the normal subgroup $J_{3,27} \times J_{3,27}$.*

Sketch of proof. By Proposition 6.3 and Lemma 2.9,

$$J_{2,12} \times J_{2,12} \times J_{2,12} \times J_{2,12} \leq N \leq M \times M \leq J_{2,11} \times J_{2,11}.$$

The proof can be achieved by analyzing the G-module $A_2 \times A_2$ and taking into account (4). ∎

6.11. THEOREM. *If $N \not\geq N_3$ then N is one of the following subgroups (where A_1 is as in* 6.10.2):

Subgroup	Description	Index	Submodule of $A_1 \times A_1$
$J_{3,30}$	$\langle J_{3,27} \times J_{3,27}, (y, 1, y, 1), (x^2, x^2)\rangle$	2^{14}	$\langle(\zeta, \zeta), (\psi, \psi)\rangle$
$J_{3,31}$	$\langle J_{3,27} \times J_{3,27}, (x^2, x^2)\rangle$	2^{15}	$\langle(\psi, \psi)\rangle$
$J_{3,32}$	$\langle J_{3,27} \times J_{3,27}, (y, 1, y, 1)(x^2, x^2)\rangle$	2^{15}	$\langle(\zeta\psi, \zeta\psi)\rangle$
$J_{3,33}$	$\langle J_{3,27} \times J_{3,27}, (y, 1, 1, 1)(x^2, x^2), (1, 1, y, 1)(x^2, x^2)\rangle$	2^{14}	$\langle(\zeta\psi, \psi), (\psi, \zeta\psi)\rangle$
$J_{3,34}$	$\langle J_{3,27} \times J_{3,27}, (y, 1, y, 1), (x^2, 1), (1, x^2)\rangle$	2^{13}	$\langle(\zeta, \zeta), (\psi, 1), (1, \psi)\rangle$
$J_{3,35}$	$\langle J_{3,27} \times J_{3,27}, (y, 1, y, 1)(x^2, 1), (y, 1, y, 1)(1, x^2)\rangle$	2^{14}	$\langle(\zeta\psi, \zeta), (\zeta, \zeta\psi)\rangle$
$J_{3,36}$	$\langle J_{3,27} \times J_{3,27}, (x^2, 1), (1, x^2)\rangle$	2^{14}	$\langle(\psi, 1), (1, \psi)\rangle$
$J_{3,37}$	$\langle J_{3,27} \times J_{3,27}, (y, 1, 1, 1)(1, x^2), (1, 1, y, 1)(x^2, 1)\rangle$	2^{14}	$\langle(\zeta, \psi), (\psi, \zeta)\rangle$
$J_{3,38}$	$\langle J_{3,27} \times J_{3,27}, (1, 1, y, 1)(1, x^2), (y, 1, 1, 1)(x^2, 1)\rangle$	2^{14}	$\langle(1, \zeta\psi), (\zeta\psi, 1)\rangle$
$J_{3,39}$	$\langle J_{3,27} \times J_{3,27}, (y, 1, 1, 1)(1, x^2),$ $(1, 1, y, 1)(x^2, 1), (y, 1, 1, 1)(x^2, 1)\rangle$	2^{13}	$\langle(\zeta, \psi), (\psi, \zeta), (\zeta\psi, 1)\rangle$

Proof. By Lemmas 6.4 and 6.9, M equals $J_{2,11}, J_{2,13}$, or $J_{2,14}$. Since $J_{2,13}, J_{2,14} \leq J_{2,11}$, by Lemma 6.10.2, we have to analyze $(J_{2,11} \times J_{2,11})/(J_{3,27} \times J_{3,27})$. Setting $\zeta = (y, 1)J_{3,27}$ and $\psi = x^2 J_{3,27}$, observing that $x^2 = (caca, acac)$, and recalling Lemma 6.10.1. and (4), we have

$$\begin{aligned} A_1 \equiv J_{2,11}/J_{3,27} &= \frac{J_{2,11}/(J_{2,12} \times J_{2,12})}{J_{3,27}/(J_{2,12} \times J_{2,12})} \\ &= \frac{\langle(\bar{y}, 1), (1, \bar{y}), (1, (\bar{c}\bar{a})^4), (\bar{c}\bar{a}\bar{c}\bar{a}, \bar{a}\bar{c}\bar{a}\bar{c})\rangle}{\langle((\bar{c}\bar{a})^4, (\bar{c}\bar{a})^4), (\bar{y}, \bar{y}), ((\bar{c}\bar{a})^4, \bar{y})\rangle} \\ &= \langle\zeta, \psi\rangle = C_2 \times C_2. \end{aligned}$$

The action of G on

$$A_1 \times A_1 = \langle(\zeta, 1), (\psi, 1), (1, \zeta), (1, \psi)\rangle = (C_2 \times C_2) \times (C_2 \times C_2) \quad (5)$$

is simply the action of a that exchanges the coordinates (thus, the G-module (5) is $P(1) \oplus P(1)$). Note that $N_3/(J_{3,27} \times J_{3,27}) \equiv \langle(\zeta, 1), (1, \zeta)\rangle$ and a subgroup of $(J_{2,11} \times J_{2,11})/(J_{3,27} \times J_{3,27})$ is contained in $\mathrm{St}_G(4)/(J_{3,27} \times J_{3,27})$ if and only if it contained in $\langle(\zeta, 1), (1, \zeta)\rangle$. Thus, to end the proof, we are only left to determine all the G-invariant (actually $\langle a\rangle$-invariant) submodules of (5) which do not contain nor are contained in $\langle(\zeta, 1), (1, \zeta)\rangle$: their list is as in the statement. ∎

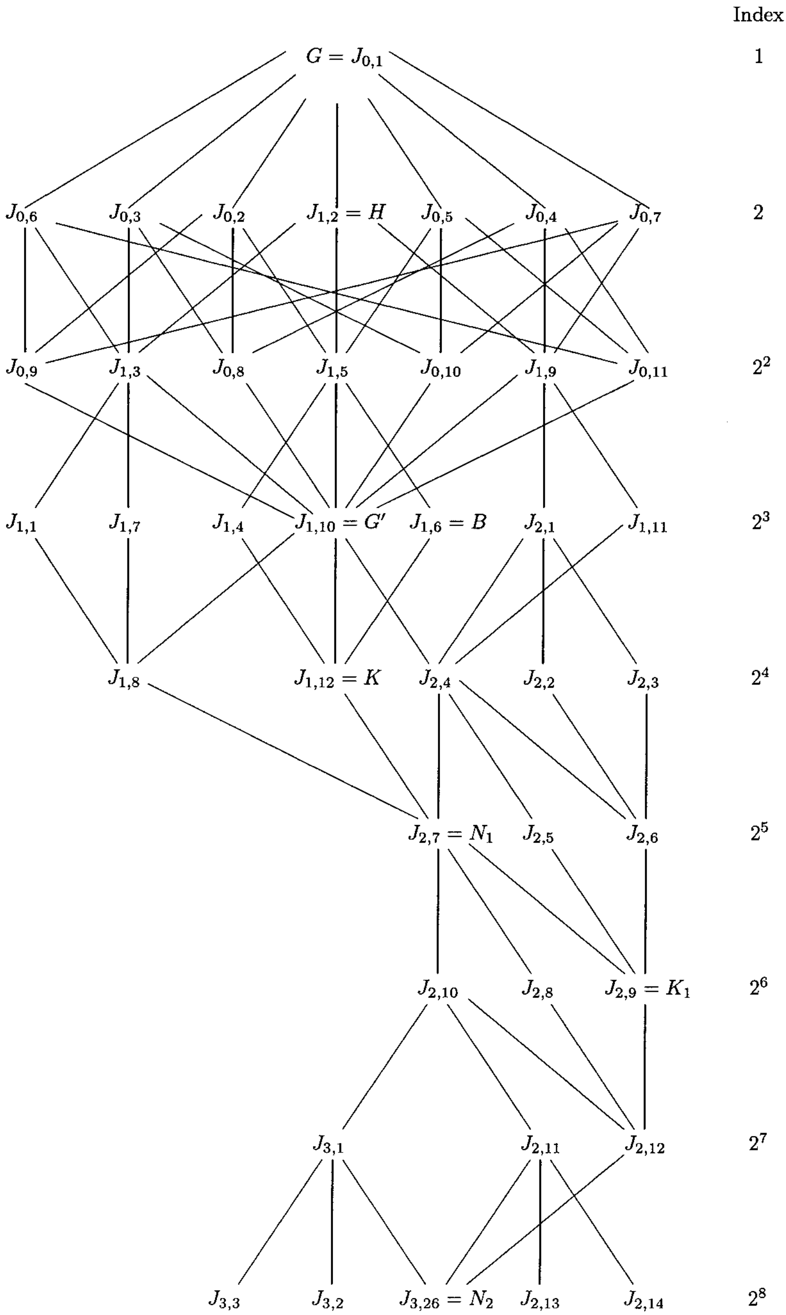

FIG. 1. The top of the lattice of normal subgroups of G.

6.12. Observing that $[G : \mathrm{St}_G(4)] = 2^{12}$, all normal subgroups N of G of index $[G : N] \le 2^{12}$ have been determined in Sections 3–6. Here below we present a table which counts all of them including $\mathrm{St}_G(4)$.

Index	1	2	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
Subgroups	1	7	7	7	5	3	3	3	5	5	7	5	7

7. NORMAL SUBGROUPS AT LEVEL $m \ge 4$: REDUCTION TO AN ABELIAN PROBLEM

7.1. In the last chapter of [4], the determination of the closed normal subgroups of the infinite iterated wreath product $\Gamma = \cdots \wr S_d \wr \cdots \wr S_d \wr S_d$, of the symmetric group S_d, is reduced to an abelian problem, namely to the determination of all Γ-submodules of certain Γ-modules.

We now give, in two steps, a similar reduction for the Grigorchuk group G: as in [4], the complete solution of such a reduced problem requires a deeper analysis of the corresponding G-modules.

Denote by $\mathfrak{N}_m$ the set of all normal subgroups of G contained in $\mathrm{St}_G(m)$ but not contained in $\mathrm{St}_G(m+1)$.

Step 1. For $N \in \mathfrak{N}_m$, $\phi_0^{m-3}(N)$ is normal in G, contained in $\mathrm{St}_G(3)$, but not contained in $\mathrm{St}_G(4)$. It follows that $\mathfrak{N}_m$ can be partitionned into the disjoint union $\bigcup_{h=1}^{39} \mathfrak{N}_{m,h}$ where $\mathfrak{N}_{m,h} = \{N \in \mathfrak{N}_m : \phi_0^{m-3}(N) = J_{3,h}\}$.

Step 2. For each $h = 1, \ldots, 39$, by Remark 5.5.1, $J_{3,h} \le \mathrm{St}_G(3) \le N_1$, so that $J_{3,h} \times \cdots \times J_{3,h}$ (2^{m-3} times) is a subgroup of G (this is no longer true, in general, if we use ϕ_0^{m-i} for $i = 0, 1, 2$). Then, as in Lemma 2.9 and Theorem 5.12, for $N \in \mathfrak{N}_{m,h}$,

$$\underbrace{[J_{3,h}, K] \times \cdots \times [J_{3,h}, K]}_{2^{m-3}} \le N \le \underbrace{J_{3,h} \times \cdots \times J_{3,h}}_{2^{m-3}}.$$

The quotient $M_h := J_{3,h}/[J_{3,h}, K]$ is *abelian*, as $J_{3,h} \le K$: we call it a *fundamental G-module*; setting

$$\mathfrak{M}_{m,h} = \underbrace{M_h \times \cdots \times M_h}_{2^{m-3}}$$

we clearly have *a bijection* between $\mathfrak{N}_{m,h}$ and the set of *all* G-submodules of $\mathfrak{M}_{m,h}$ *with surjective projection on each factor.*

7.2. As a consequence, we are left with the following problems:

(i) classification of all the *fundamental G-modules* M_h and their G-submodules;

(ii) determination of all G-submodules of the $\mathfrak{M}_{m,h}$'s.

7.3. With a slight variation of the arguments in 7.1 (namely considering also the projections $\phi_0^{m-1}(N)$ and $\phi_0^{m-2}(N)$ at levels 1 and 2: we omit the details) one can show that in 7.2 the number of fundamental G-modules can be reduced to 12: the set $\{J_{3,1}, \ldots, J_{3,39}\}$ may be replaced by $\{J_{1,12}\} \bigcup \{J_{2,h} : h = 7, 8, 10, 11, 13, 14\} \bigcup \{J_{3,h} : h = 1, 2, 3, 4, 5\}$.

The analysis of the third level in Section 6 and the congruence property in 5.12 suggest that an even further reduction of the number of modules is possible, namely that the determination of all normal subgroups of level $m \geq 4$ can be achieved by analyzing (as in 7.2) only three fundamental G-modules. We believe that two of these are N_{m-1}/N_m and $\mathrm{St}_G(m)/K_{m-1}$ as in Section 6 ($m = 3$). Observe that

$$(6) \qquad N_{m-1}/N_m \cong \frac{\mathrm{St}_G(m+1)}{K_m} \cong P(m-1) \oplus P(m-2), \qquad m \geq 3.$$

7.4. Recall from [2] that the G-modules $P(m)$ are *uniserial*. In virtue of this and (6) we pose the following general:

Problem. Let Γ be a group and et P_1 and let P_2 be uniserial Γ-modules. Determine all the Γ-subspaces of their direct sum $P_1 \oplus P_2$, that is, of $P_1 \times P_2$ with the diagonal action of Γ: $(v, u)^g = (v^g, u^g)$, $v \in P_1, u \in P_2$.

7.5. After a preliminary version of this paper was circulated, we received from L. Bartholdi his preprint [1] where, among other things, close calculations and results are obtained with Lie-algebraic methods [2] leading to a description of all normal subgroups and computation of the *normal subgroup growth* of G.

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REFERENCES

1. L. Bartholdi, "Lie Algebras and Growth in Branch Groups," preprint Math. GR/0101222, 2001.
2. L. Bartholdi and R. I. Grigorchuk, Lie methods in Growth of Groups and Groups of finite width, in "Computational and Geometric Aspects of Cambridge Univ. Press, Modern Algebra" (M. Atkinson et al., Eds.), London Math. Soc. Ser., Vol. 275 1–27.
3. L. Bartholdi and R. I. Grigorchuk, On parabolic subgroups and Hecke algebras of some fractal groups, to appear.
4. H. Bass, M. V. Otero-Espinar, D. Rockmore, and C. Tresser, "Cyclic Renormalization and Automorphism Groups of Rooted Trees," Lecture Notes in Mathematics, Vol. 1621, Springer-Verlag, Berlin, 1996.

5. R. I Grigorchuk, On Burnside's problem on periodic groups, *Funct. Anal. Appl.* **14** (1980), 41–43.
6. R. I. Grigorchuk, Degrees of growth of finitely generated groups, and the theory of invariant means, *Math. USSR Izv.* **25** (1985).
7. R. I. Grigorchuk, Just infinite branch groups, *Progr. Math.* **184** (2000), 121–179.
8. P. de la Harpe, "Topics in Geometric Group Theory," Univ. of Chicago Press, Chicago, 2000.
9. A. V. Rozhkov, Lower central series of a group of tree automorphisms, *Mat. Zametki* **60** (1966), 225–237 (translated in *Math. Notes* **60** (1997), 165–174.).